Reinterpretation of the Auxiliary Differential Equation Method for FDTD

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***Abstract*—The auxiliary differential equation method used for treating Lorentz media with the FDTD algorithm is reinterpreted to reduce computer memory requirements while maintaining full time synchronism. This new formulation enables us to save up to 20% of computer memory when compared to the previous approach. The validity of our formulation is proved analytically by comparing the resulting equations with those from the previous method. We also employ this approach to Debye media and show that no disadvantage arises in terms of memory requirement.**

***Index Terms*—Auxiliary differential equation method, dispersive medium, finite difference time domain method.**

I. Introduction

The FINITE-DIFFERENCE time-domain (FDTD) method is well known as one of the most powerful numerical methods for solving electromagnetic problems. In the past, several attempts have been made to extend the application range of the FDTD method toward electrically dispersive media, and, so far, three methods have been proposed: the recursive convolution (RC) method, the auxiliary differential equation (ADE) method, and the Z-transform (ZT) method [1]–[6]. RC has least memory requirements, but ADE and ZT offer more accurate results. Recently, the accuracy of RC has been improved to a level which comes close to that of ADE and ZT by employing the modified technique of piecewise-linear recursive convolution (PLRC) [7]. At the same time, however, memory requirements for ADE have also been reduced to a similar amount as that used by PLRC [8]. Although it has been pointed out that different kinds of dispersive media require different treatments in ADE but allow a unified treatment in RC and PLRC, the arithmetic used in ADE is much easier [7]–[9].

With respect to ADE, three new and efficient types of algorithms for multi-pole Lorentz and Debye media were presented in [8]: the Debye ADE synchronized (DADES), the Lorentz ADE synchronized (LADES), and the Lorentz ADE partially synchronized (LADEP) scheme. In DADES, time synchronism is fully maintained by adopting the semi-implicit scheme in the finite-difference expression. In LADES, the full time synchronism is satisfied; however, an additional back-storage of the electric field components is required. In LEDEP, the back-stored electric field components are not necessary, but the computation is less accurate than that of LEDES due to the partial time synchronism.

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In this letter, we reinterpret the approach of [8] to reduce the required computer memory while keeping full time synchronism. This reinterpretation results in simpler FDTD expressions, and the memory requirements are reduced to the same degree as needed for partial time synchronism. The validity of the reinterpretation is proved analytically by comparing the equations obtained in our approach with those of the previous one. Finally, we also mention the validity of our formulation with respect to Debye media showing that no disadvantage in memory requirement arises when compared to [8].

II. Reinterpretation of the Auxiliary Differential Equation Method for Lorentz Media

Maxwell's equations in time domain are described by

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{1}$$

$$\nabla \times \vec{H} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}. \tag{2}$$

The permittivity of a Lorentz medium with P pole pairs is given in the frequency domain by

$$\varepsilon_\omega = \varepsilon_\infty + \sum_p^P \frac{\Delta\varepsilon A_p \omega_p^2}{\omega_p^2 - 2i\omega\delta_p - \omega^2} \tag{3}$$

where ω_p is the pth resonant frequency, A_p is the pole amplitude, δ_p is the damping factor, and $\Delta\varepsilon = \varepsilon_s - \varepsilon_\infty$, where ε_s and ε_∞ are the static and infinite frequency permittivities, respectively. The time dependence is described using the $e^{-i\omega t}$ convention. Ampere's law in the frequency domain then becomes

$$\nabla \times \vec{H}_\omega = \sigma \vec{E}_\omega - i\omega\varepsilon_\infty \vec{E}_\omega - i\omega \sum_p^P \vec{Q}_{\omega p} \tag{4}$$

with

$$\vec{Q}_{\omega p} = \frac{\Delta\varepsilon A_p \omega_p^2}{\omega_p^2 - 2i\omega\delta_p - \omega^2} \vec{E}_\omega. \tag{5}$$

Taking the inverse Fourier transform of (4) and (5), and introducing the discrete time steps $t = n\Delta t$, the FDTD notation of (4) and (5) centered at time step n follows as

$$\vec{E}^{\,n+1} = \frac{2\varepsilon_\infty - \sigma\Delta t}{2\varepsilon_\infty + \sigma\Delta t} \vec{E}^{\,n} + \frac{2\Delta t}{2\varepsilon_\infty + \sigma\Delta t} \left(\nabla \times \vec{H}^{\,n+1/2}\right) - \frac{2}{2\varepsilon_\infty + \sigma\Delta t} \sum_p^P \left(\vec{Q}_p^{\,n+1} - \vec{Q}_p^{\,n}\right) \tag{6}$$

$$\vec{Q}_p^{\,n+1} = \frac{\delta_p\Delta t - 1}{\delta_p\Delta t + 1} \vec{Q}_p^{\,n-1} + \frac{2 - (\Delta t)^2\omega_p^2}{\delta_p\Delta t + 1} \vec{Q}_p^{\,n} + \frac{(\Delta t)^2 \Delta\varepsilon A_p \omega_p^2}{\delta_p\Delta t + 1} \vec{E}^{\,n}. \tag{7}$$

Furthermore, the FDTD description of equation (1) centered at time step n is

$$\vec{H}^{\,n+1/2} = \vec{H}^{\,n-1/2} - \frac{\Delta t}{\mu}\left(\nabla \times \vec{E}^{\,n}\right). \tag{8}$$

The values of the electromagnetic field components are updated by an explicit three-steps algorithm in the order of (8), (7), and (6). Note that these equations maintain the full time synchronism. The components needed to be stored are the one-time-step previous values $\vec{H}^{\,n-1/2}$, $\vec{E}^{\,n}$, $\vec{Q}_p^{\,n}$, and the additional back-stored $\vec{Q}_p^{\,n-1}$ as shown in the schematic flowchart of Fig. 1(a).

In the previous approach of LADES [8], the corresponding expressions to (4) and (5) were given as

$$\nabla \times \vec{H}_\omega = \sigma \vec{E}_\omega - i\omega\varepsilon_\infty + \sum_p^P \vec{J}_{\omega p} \tag{9}$$

$$\vec{J}_{\omega p} = \frac{-i\omega\Delta\varepsilon A_p \omega_p^2}{\omega_p^2 - 2i\omega\delta_p - \omega^2}. \tag{10}$$

The FDTD notation with the full time synchronism is then derived as

$$\begin{aligned}\vec{E}^{\,n+1} = {} & \frac{p_m}{4}\sum_p^P \left[\frac{\Delta t \Delta\varepsilon A_p \omega_p^2}{\delta_p \Delta t + 1}\right] \vec{E}^{\,n-1} + p_e \vec{E}^{\,n} \\ & + p_m\left(\nabla \times \vec{H}^{\,n+1/2}\right) \\ & - \frac{p_m}{2}\sum_p^P \left[\left(\frac{2-(\Delta t)^2\omega_p^2}{\delta_p\Delta t + 1} + 1\right)\vec{J}_p^{\,n} \right. \\ & \left. + \frac{\delta_p \Delta t - 1}{\delta_p \Delta t + 1}\vec{J}_p^{\,n-1}\right]\end{aligned} \tag{11}$$

$$\begin{aligned}\vec{J}_p^{\,n+1} = {} & \frac{\delta_p\Delta t - 1}{\delta_p \Delta t + 1}\vec{J}_p^{\,n-1} + \frac{2-(\Delta t)^2\omega_p^2}{\delta_p\Delta t + 1}\vec{J}_p^{\,n} \\ & + \frac{\Delta t \Delta\varepsilon A_p\omega_p^2}{2(\delta_p\Delta t + 1)}\left(\vec{E}^{\,n+1} - \vec{E}^{\,n-1}\right),\end{aligned} \tag{12}$$

where

$$p_e = \frac{2\varepsilon_\infty - \sigma\Delta t}{2\varepsilon_\infty + \sigma\Delta t + \frac{1}{2}\sum_p^P \frac{(\Delta t)^2\Delta\varepsilon A_p\omega_p^2}{\delta_p\Delta t + 1}} \tag{13}$$

$$p_m = \frac{2\Delta t}{2\varepsilon_\infty + \sigma\Delta t + \frac{1}{2}\sum_p^P \frac{(\Delta t)^2\Delta\varepsilon A_p\omega_p^2}{\delta_p\Delta t + 1}}. \tag{14}$$

The electromagnetic field components are updated according to (8), (12), and (11). The components to be stored are $\vec{H}^{\,n-1/2}$, $\vec{E}^{\,n}$, $\vec{J}_p^{\,n}$, together with the two back-stored components $\vec{J}_p^{\,n-1}$ and $\vec{E}^{\,n-1}$, as shown in Fig. 1(b).

The amount of memory required by the computation flow described by Fig. 1(a) is less than the one described by Fig. 1(b). A maximum of 20% memory (in case of $P = 1$) can be saved while satisfying the full time synchronic scheme. This corresponds to the amount of memory used for the partial synchronic scheme [8]. The simpler expressions of the equations and the smaller number of arrows in the flowchart of Fig. 1(a) indicate the reduced number of arithmetic operations required in our approach.

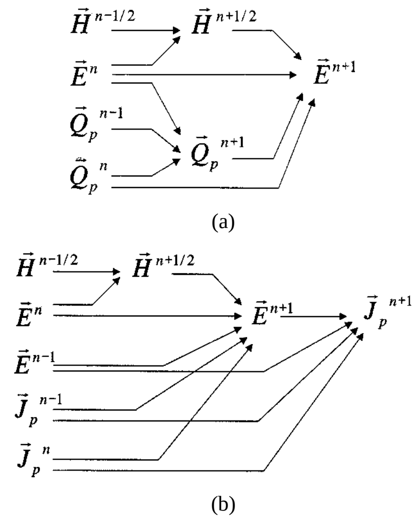

Fig. 1. Schematic flowcharts. (a) Reformulated approach for Lorentz media, (b) previous approach LADES.

The identity of (6) and (11) is easily shown.

The relation connecting (4) and (9) is

$$-i\omega\vec{Q}_{\omega p} = \vec{J}_{\omega p}. \tag{15}$$

In the FDTD description, we use

$$\frac{\vec{Q}_p^{\,n+1} - \vec{Q}_p^{\,n}}{\Delta t} = \frac{\vec{J}_p^{\,n+1} + \vec{J}_p^{\,n}}{2}. \tag{16}$$

Substituting (16) into (6), and (12) into the resulting equation, (11) can be derived. Hence, the electromagnetic field components computed by (6) to (8) are equivalent to those from the previous formulation.

III. Reinterpretation of the Auxiliary Differential Equation Method for Debye Media

In the following, a similar approach is applied to Debye media. The permittivity of a P pole Debye media is given in the frequency domain by

$$\varepsilon_\omega = \varepsilon_\infty + \sum_p^P \frac{\Delta\varepsilon A_p}{1 - i\omega\tau_p} \tag{17}$$

where τ_p is the relaxation time of pth pole. The FDTD description corresponding to (7) is

$$\vec{Q}_p^{\,n+1} = \vec{Q}_p^{\,n-1} - \frac{2\Delta t}{\tau_p}\vec{Q}_p^{\,n} + \frac{2\Delta t\Delta\varepsilon A_p}{\tau_p}\vec{E}^{\,n} \tag{18}$$

having the discretization center at time step n. For the computation, (6) and (8) are used too. The schematic flowchart of this approach is shown in Fig. 2(a) which is completely the same as Fig. 1(a) describing the flowchart for treating Lorentz media.

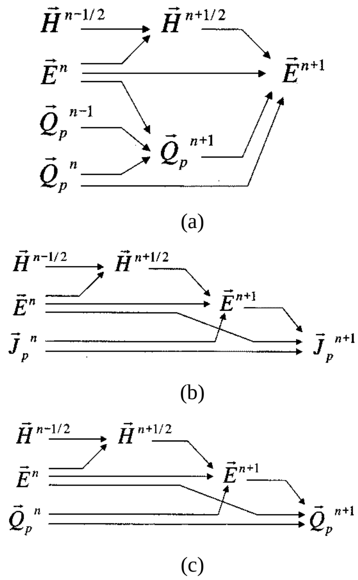

Fig. 2. Schematic flowcharts. (a) Reformulated approach for Debye media with the discretization center at time step n, (b) previous approach DADES, and (c) reformulated approach for Debye media with the discretization center at time step $n + 1/2$.

In the approach of DADES [8], the corresponding equations to (11) and (12) are

$$\vec{E}^{\,n+1} = p_e\vec{E}^{\,n} + p_m\left(\nabla\times\vec{H}^{\,n+1/2}\right) - 2p_m\sum_p^P\left[\frac{\tau_p}{2\tau_p+\Delta t}\,\vec{J}_p^{\,n}\right] \tag{19}$$

$$\vec{J}_p^{\,n+1} = \frac{2\tau_p-\Delta t}{2\tau_p+\Delta t}\,\vec{J}_p^{\,n} + \frac{2\Delta\varepsilon A_p}{2\tau_p+\Delta t}\left(\vec{E}^{\,n+1}-\vec{E}^{\,n}\right) \tag{20}$$

where

$$p_e = \frac{2\varepsilon_\infty - \sigma\Delta t + 2\displaystyle\sum_p^P\frac{\Delta t\Delta\varepsilon A_p}{2\tau_p+\Delta t}}{2\varepsilon_\infty + \sigma\Delta t + 2\displaystyle\sum_p^P\frac{\Delta t\Delta\varepsilon A_p}{2\tau_p+\Delta t}} \tag{21}$$

$$p_m = \frac{2\Delta t}{2\varepsilon_\infty + \sigma\Delta t + 2\displaystyle\sum_p^P\frac{\Delta t\Delta\varepsilon A_p}{2\tau_p+\Delta t}}. \tag{22}$$

The identity of (6) and (19) is proven by substituting first (20) into (16) and then the resulting equation into (6). The calculation flow of (19) and (20) is shown in Fig. 2(b). In case of Fig. 2(a), the number of components to be stored is higher than that for the case of Fig. 2(b); however, the simple expressions and the possibility of using (6) for both Lorentz and Debye media are attractive.

If the discrete expression of $\vec{Q}_p^{\,n+1}$ is centered at time step $n + 1/2$, the FDTD description corresponding to (18) becomes

$$\vec{Q}_p^{\,n+1} = \frac{2\tau_p-\Delta t}{2\tau_p+\Delta t}\,\vec{Q}_p^{\,n} + \frac{\Delta t\Delta\varepsilon A_p}{2\tau_p+\Delta t}\left(\vec{E}^{\,n+1}+\vec{E}^{\,n}\right). \tag{23}$$

Substituting (23) to (6), $\vec{E}^{\,n+1}$ is obtained as

$$\vec{E}^{\,n+1} = c_e\vec{E}^{\,n} + c_m\left(\nabla\times\vec{H}^{\,n+1/2}\right) + 2c_m\sum_p^P\left[\frac{1}{2\tau_p+\Delta t}\,\vec{Q}_p^{\,n}\right] \tag{24}$$

$$c_e = \frac{2\varepsilon_\infty - \sigma\Delta t - 2\displaystyle\sum_p^P\frac{\Delta t\Delta\varepsilon A_p}{2\tau_p+\Delta t}}{2\varepsilon_\infty + \sigma\Delta t + 2\displaystyle\sum_p^P\frac{\Delta t\Delta\varepsilon A_p}{2\tau_p+\Delta t}}. \tag{25}$$

The equations are very similar to those given by (19) to (22) and the data flow shown in Fig. 2(c) is of the same type as the one shown in Fig. 2(b). The required computer memory and the number of arithmetic operations are equivalent to the previous method.

IV. Conclusion

The ADE method for treating Lorentz media with the finite time domain algorithm has been reinterpreted. This reinterpretation is able to reduce up to 20% of the required amount of computer memory needed to store the electromagnetic field components. It does so by using simpler equations that reduce the overall number of arithmetic operations when compared to the previous method. We also employed this approach to Debye media and derived two different sets of equations. One gives us the advantage of using simpler and identical expressions for treating both Lorentz and Debye media, although the requirement of computer memory is higher than in the previous approach. The other presents us with expressions similar to those of the previous method, but the required amount of memory as well as the number of arithmetic operations is the same.

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